Recursive solution for beam dynamics of periodic focusing channels

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We present recursive analysis for beam dynamics of periodic focusing channels based on the Fourier coefficients of the focusing function. Formulas for orbit stability and the envelope function are derived. The results should be useful for numerical calculation and for developing analytical understanding of channels employing extended focusing elements. Applications to muon ionization cooling channels are discussed.

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I. INTRODUCTION

Periodic focusing channels serve important functions in particle accelerators, for example, as a beam transport line between two accelerator sections or as a special beam manipulation station for cooling [1–4]. In designing a channel, it is necessary to have an efficient method to compute the beam properties starting from the arrangement of the focusing elements. For high-energy accelerators employing quadrupole magnets, the focusing elements can usually be regarded as simple lenses. The beam dynamics analysis can then be based on the manipulation of a few matrices [1]. However, this approach does not work for other cases; an important example is the solenoidal focusing channel proposed recently for ionization cooling of muon beams [5], where the solenoidal field extends the whole length of the channel. In this paper we present a systematic treatment of the beam dynamics in general periodic channels using the perturbation technique, derive formulas for orbit stability and envelope function, and discuss applications to the solenoidal focusing channels considered for muon cooling. Some aspects of our treatment exist in the literature. However, our analysis is more systematic, yields results, and connects to the established results on Hill's equation, studied by Hill in his 1877 memoir on the motion of the lunar perigee [6] and continued by others [7–9]. *As far as we are aware, however, these results have not been properly appreciated in the beam physics community until now*.

Although this paper is on beam dynamics in accelerators, it deals essentially with the stability and oscillation amplitude of an anharmonic oscillator with periodic restoring force. An anharmonic oscillator is a basic model for numerous physical systems. Therefore, the presented mathematical technique and results should be useful beyond linear optics in accelerators. Knowledge of beam optics is not necessary to appreciate these results, and we made an effort to be pedagogical.

II. THEORY

A. Standard formalism

Let us briefly summarize the standard approach for beam dynamics analysis of a periodic focusing channel. The equation for transverse motion is [1]

$$\frac{d^2x(s)}{ds^2}+K(s)x(s)=0. \tag{1}$$

Here x is the transverse displacement of a particle and s is its longitudinal coordinate. $K(s)$, referred to as the focusing function, describes the arrangement of focusing elements and is assumed to be periodic with a period L: $K(s+L)=K(s)$. For a particle of charge q and longitudinal momentum p_s, $K(s)=qB_1(s)/p_s$ in a quadrupole channel of field gradient $B_1(s)$ and $K(s)=[qB_s(s)/2p_s]^2$ in a solenoid channel of on-axis field $B_s(s)$. The solution of Eq. (1) is conveniently parametrized in the Floquet form $x(s)=\sqrt{\epsilon\beta(s)}\cos[\psi(s)+\phi]$. Here ϵ and ϕ are constants specifying a particular particle within a beam, $\beta(s)$ is a periodic function referred to as the envelope function, and $\psi(s)=\int_0^s d\bar{s}/\beta(\bar{s})$ is the phase advance. The phase advance in one period $\mu=\psi(L)$ and the envelope function β are quantities of fundamental importance in beam physics: μ reflects the basic frequency of the system and $\beta(s)$ gives the scale and the s-dependent profile of the beam size.

Computations of μ, β, and other quantities are usually carried out by constructing the transfer matrix R defined as follows:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix}=R(s)\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix}. \tag{2}$$

The elements of R are

$$R(s)=\begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix}. \tag{3}$$

Here, the prime indicates differentiation with respect to s. The "cosinelike" orbit $C(s)$ and the "sinelike" orbit $S(s)$ are special solutions of Eq. (1) satisfying the initial conditions $C(0)=1$, $C'(0)=0$, $S(0)=0$, and $S'(0)=1$. In terms of ψ, β, and $\alpha=-\beta'(s)/2$, the transfer matrix is

$$R(s)=\begin{bmatrix}\sqrt{\frac{\beta(s)}{\beta(0)}}[\cos\psi+\alpha(0)\sin\psi] & \sqrt{\beta(0)\beta(s)}\sin\psi\\ \frac{\alpha(0)-\alpha(s)}{\sqrt{\beta(0)\beta(s)}}\cos\psi-\frac{1+\alpha(0)\alpha(s)}{\sqrt{\beta(0)\beta(s)}}\sin\psi & \sqrt{\frac{\beta(0)}{\beta(s)}}[\cos\psi-\alpha(s)\sin\psi]\end{bmatrix}. \tag{4}$$

Let the trace of the one-period matrix be

$$\Delta\equiv \mathrm{Tr}\,R(L)=C(L)+S'(L). \tag{5}$$

The quantity Δ is important since it enters into the stability criteria of the motion in a channel. The motion is stable if and only if [10]

$$|\Delta|\leqslant 2. \tag{6}$$

For stable motion, the one-period phase advance μ is then given by

$$\cos\mu=\frac{\Delta}{2}. \tag{7}$$

The value of the envelope function at $s=0$ is given by

$$\beta(0)=\frac{1}{\sin\mu}R_{12}(L)=\frac{1}{\sin\mu}S(L). \tag{8}$$

The value of $\beta(s)$ at an arbitrary s can be obtained by constructing a transfer matrix from the point s to $s+L$.

Explicit calculation is greatly simplified when the focusing elements can be considered as simple lenses. The one-period transfer matrix can then be obtained by multiplying a small number of simple matrices representing focusing elements and free spaces. Unfortunately, this simplification is not applicable for channels consisting of extended focusing elements, as in the case of solenoidal focusing.

B. Recursive solution

To develop a more general approach, we begin by expanding the focusing function in Fourier series as follows:

$$\left(\frac{L}{\pi}\right)^2 K(s)=\sum_{n=-\infty}^{\infty}\vartheta_n e^{i2n\pi s/L}=\vartheta_0+\tilde{\vartheta}(\varsigma). \tag{9}$$

Here

$$\varsigma=\pi s/L, \tag{10}$$

is the scaled s variable with period π. Note that ϑ_n and ς are dimensionless. Since $K(s)$ is real, $\vartheta_n^*=\vartheta_{-n}$, where ϑ_n^* is the complex conjugate of ϑ_n.[1] The constant part ϑ_0 is important since $\sqrt{\vartheta_0}$ is the fundamental frequency. The variable part, denoted by the symbol $\tilde{\vartheta}(\varsigma)$, may be regarded as small in the sense that its average vanishes, suggesting that Eq. (1) may be solved iteratively by writing it in the following form [9]:

$$\frac{d^2x}{d\varsigma^2}+\vartheta_0 x=-\tilde{\vartheta}(\varsigma)x. \tag{11}$$

The solution of the form $x=\Sigma_{k=0}^{\infty}x_k$ is determined from the recursive relations

$$\ddot{x}_0+\vartheta_0 x_0=0 \quad\text{and}\quad \ddot{x}_k+\vartheta_0 x_k=-\tilde{\vartheta}x_{k-1}\quad\text{for}\quad k\geqslant 1, \tag{12}$$

where the dots indicate differentiation with respect to ς. Initial conditions for $x_0(\varsigma)$ are chosen to be the same as those for $x(\varsigma)$, while $x_k(0)=\dot{x}_k(0)=0$ for $k>0$. The two independent solutions $u(\varsigma)=\Sigma_{k=0}^{\infty}u_k(\varsigma)$ and $v(\varsigma)=\Sigma_{k=0}^{\infty}v_k(\varsigma)$ satisfying the initial conditions $u(0)=1$ and $\dot{u}(0)=0$ and $v(0)=0$ and $\dot{v}(0)=1$, respectively, are given by the following recursive relations:

$$u_0(\varsigma)=\cos(\sqrt{\vartheta_0}\varsigma)\quad\text{and}\quad v_0(\varsigma)=\frac{\sin(\sqrt{\vartheta_0}\varsigma)}{\sqrt{\vartheta_0}}, \tag{13}$$

$$u_k(\varsigma)=-\int_0^{\varsigma}v_0(\varsigma-\bar{\varsigma})\tilde{\vartheta}(\bar{\varsigma})u_{k-1}(\bar{\varsigma})d\bar{\varsigma}, \tag{14}$$

$$v_k(\varsigma)=-\int_0^{\varsigma}v_0(\varsigma-\bar{\varsigma})\tilde{\vartheta}(\bar{\varsigma})v_{k-1}(\bar{\varsigma})d\bar{\varsigma}. \tag{15}$$

The solutions in the s variable are then given by a simple rescaling

$$C(s)=u(\varsigma)\quad\text{and}\quad S(s)=\frac{L}{\pi}v(\varsigma). \tag{16}$$

This completes the discussion of the general procedure for computing the R matrix and other beam dynamics quantities recursively, based on the Fourier coefficients of the focusing function.

C. Stability criteria

For more explicit results, let us work out the recursive expansion to some low orders. Consider first the trace

$$\Delta=u(\pi)+\dot{v}(\pi)=\Delta_0+\Delta_1+\Delta_2+\Delta_3+,\ldots. \tag{17}$$

[1] In literature for Hill's equation, it is often assumed that $K(s)$ is an even function in s, implying that ϑ_n are real. We will not need this assumption here.

The zeroth-order term Δ_0 is easy to get from Eq. (13) as

$$\Delta_0 = 2\cos(\sqrt{\vartheta_0}\pi), \tag{18}$$

while Δ_k is, in view of Eqs. (14),(15),

$$\Delta_k = -\int_0^{\pi} d\varsigma\, \tilde{\vartheta}(\varsigma)[v_0(\pi-\varsigma)u_{k-1}(\varsigma) + u_0(\pi-\varsigma)v_{k-1}(\varsigma)]. \tag{19}$$

The first-order term Δ_1 vanishes because it can be reduced to an integral over one period of the function $\tilde{\vartheta}$, i.e.,

$$\Delta_1 = -\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}}\int_0^{\pi}\tilde{\vartheta}(\varsigma)d\varsigma = 0. \tag{20}$$

The second-order term Δ_2 is equal to the integral

$$\begin{aligned}
&\int_0^{\pi} d\varsigma\, \tilde{\vartheta}(\varsigma)\int_0^{\varsigma} d\bar{\varsigma}\, \tilde{\vartheta}(\bar{\varsigma}) v_0(\varsigma-\bar{\varsigma})[v_0(\pi-\varsigma)u_0(\bar{\varsigma}) + u_0(\pi-\varsigma)v_0(\bar{\varsigma})] \\
&= \int_0^{\pi} d\varsigma \int_0^{\varsigma} d\bar{\varsigma}\, \tilde{\vartheta}(\varsigma)\tilde{\vartheta}(\bar{\varsigma}) \frac{\sin[\sqrt{\vartheta_0}(\varsigma-\bar{\varsigma})]\sin[\sqrt{\vartheta_0}(\pi-\varsigma+\bar{\varsigma})]}{\vartheta_0} \\
&= \frac{1}{\vartheta_0}\sum_{m,n=-\infty}^{\infty}\tilde{\vartheta}_m\tilde{\vartheta}_n\int_0^{\pi} d\varsigma\int_0^{\varsigma} d\bar{\varsigma}\, e^{i2n\varsigma}e^{i2m\bar{\varsigma}}\sin[\cdots]\sin[\cdots].
\end{aligned} \tag{21}$$

Here and in the following, $\tilde{\vartheta}_n$ is defined to be $\tilde{\vartheta}_n = \vartheta_n$ for $n \neq 0$ and $\tilde{\vartheta}_0 = 0$. In the Appendix we show that the terms with $m \neq -n$ in the above sum cancel out. Computing Δ_2 by collecting the contributions from the $m=-n$ terms and adding to Δ_0, we obtain the following simple expression for Δ valid to the second order:

$$\Delta = 2\cos(\sqrt{\vartheta_0}\pi) + \frac{\pi\sin\sqrt{\vartheta_0}\pi}{2\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{|\vartheta_n|^2}{\vartheta_0 - n^2} + , \ldots, . \tag{22}$$

At this point we remark that an exact expression for Δ was obtained by Hill in terms of an infinite determinant as follows [6,7]:

$$\Delta = 2 - 4\sin^2\left(\frac{\pi}{2}\sqrt{\vartheta_0}\right)D. \tag{23}$$

Here D is an infinite determinant, known as Hill's determinant, given by [11]

$$D = \begin{vmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 1 & \frac{\vartheta_{-1}}{\vartheta_0-16} & \frac{\vartheta_{-2}}{\vartheta_0-16} & \frac{\vartheta_{-3}}{\vartheta_0-16} & \frac{\vartheta_{-4}}{\vartheta_0-16} & \cdots \\
\cdots & \frac{\vartheta_1}{\vartheta_0-4} & 1 & \frac{\vartheta_{-1}}{\vartheta_0-4} & \frac{\vartheta_{-2}}{\vartheta_0-4} & \frac{\vartheta_{-3}}{\vartheta_0-4} & \cdots \\
\cdots & \frac{\vartheta_2}{\vartheta_0-0} & \frac{\vartheta_1}{\vartheta_0-0} & 1 & \frac{\vartheta_{-1}}{\vartheta_0-0} & \frac{\vartheta_{-2}}{\vartheta_0-0} & \cdots \\
\cdots & \frac{\vartheta_3}{\vartheta_0-4} & \frac{\vartheta_2}{\vartheta_0-4} & \frac{\vartheta_1}{\vartheta_0-4} & 1 & \frac{\vartheta_{-1}}{\vartheta_0-4} & \cdots \\
\cdots & \frac{\vartheta_4}{\vartheta_0-16} & \frac{\vartheta_3}{\vartheta_0-16} & \frac{\vartheta_2}{\vartheta_0-16} & \frac{\vartheta_1}{\vartheta_0-16} & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix}. \tag{24}$$

This determinant can be expanded in a power series in $\{\vartheta_n\}$. Inserting the result into Eq. (23), one obtains Δ as a power series in $\{\vartheta_n\}$, which is in fact identical to the recursive solution found above. Indeed, it is not difficult to verify the second-order result Eq. (22) by expanding the Hill's determinant. It turns out that explicit expressions of Δ_k for $k>2$ are more easily obtained by expanding the Hill's determinant than by carrying out the integrals in the recursive method. The third-order term is [9]

$$\Delta_3=\frac{\pi\sin\sqrt{\vartheta_0}\pi}{4\sqrt{\vartheta_0}}\sum_{m,n=1}^{\infty}\frac{\mathrm{Re}(\vartheta_m\vartheta_n\vartheta_{m+n}^*)(m^2+n^2+mn-3\vartheta_0)}{(\vartheta_0-m^2)(\vartheta_0-n^2)[\vartheta_0-(m+n)^2]}. \tag{25}$$

Here the symbol Re implies taking the real part.

We note the important role of the dimensionless frequency $\sqrt{\vartheta_0}$ in determining the general character of the beam transport properties in periodic channels. Equation (22) indicates that the stability condition Eq. (6) could be violated when $\sqrt{\vartheta_0}\sim n$ for any integer n, implying that the motion becomes unstable. This is referred to as the $n\pi$ *resonance* since the one-period phase advance becomes $n\pi$. The strength of the $n\pi$ resonance depends on the magnitude of the coefficient ϑ_n. From Eq. (22), assuming that the corresponding harmonics are not too weak, the stability boundaries are roughly given by

$$\sqrt{\vartheta_0}\sim n\pm\frac{1}{2}\left|\frac{\vartheta_n}{\vartheta_0}\right|+\frac{5}{16}\left|\frac{\vartheta_n}{\vartheta_0}\right|^2+,\ldots,. \tag{26}$$

The stability as a function of $\sqrt{\vartheta_0}$ can be translated to the stability as a function of other physical parameters of the system, such as the particle momentum or the rms field strength.

D. Envelope function

We now turn our attention to the envelope function $\beta(s)$. To compute $\beta(0)$ from Eqs. (8,16), we need to find $v(\pi)=\Sigma_{k=0}^{\infty}v_k(\pi)$. The zeroth-order term $v_0(\pi)$ is given in Eq. (13), and $v_1(\pi)$ can be worked out using Eq. (15) as

$$v_1(\pi)=-\sum_{n=-\infty}^{\infty}\tilde{\vartheta}_n\int_0^{\pi}d\varsigma\, e^{i2n\varsigma}\frac{\sin[\sqrt{\vartheta_0}(\pi-\varsigma)]\sin(\sqrt{\vartheta_0}\varsigma)}{\vartheta_0}$$
$$=\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{\mathrm{Re}[\vartheta_n]}{n^2-\vartheta_0}. \tag{27}$$

Adding these contributions, the envelope function at $s=0$ to first order becomes

$$\beta(0)=\frac{L}{\pi}\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}\sin\mu}\left[1+\sum_{n=1}^{\infty}\frac{\mathrm{Re}[\vartheta_n]}{n^2-\vartheta_0}+,\ldots,\right]. \tag{28}$$

The calculation of $\beta(s_0)$ at an arbitrary location s_0 is similar to the above except that we need to move the period from the interval $0\leqslant s\leqslant L$ to $s_0\leqslant s\leqslant s_0+L$. Accordingly, the Fourier coefficient ϑ_n in Eq. (9) is replaced by $\vartheta_n e^{i2n\varsigma_0}$, where $\varsigma_0=\pi s_0/L$. It then follows that the expression for $\beta(s)$ is simply obtained from the right-hand side of Eq. (28) by the replacement $\vartheta_n\to\vartheta_n e^{i2n\varsigma}$:

$$\beta(s)=\frac{L}{\pi}\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}\sin\mu}\left[1+\sum_{n=1}^{\infty}\frac{\mathrm{Re}[\vartheta_n e^{i2n\pi s/L}]}{n^2-\vartheta_0}+,\ldots,\right]. \tag{29}$$

We have also computed the second-order correction β_2. After some algebra (done with Mathematica®):

$$\beta_2(s)=\frac{L}{\pi}\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}\sin\mu}\frac{1}{4}\left\{\frac{\pi\cot(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{|\vartheta_n|^2}{n^2-\vartheta_0}+\sum_{m,n=1}^{\infty}\left[\frac{(m^2+mn+n^2-3\vartheta_0)\mathrm{Re}[\vartheta_m\vartheta_n e^{i2(m+n)\pi s/L}]}{(m^2-\vartheta_0)(n^2-\vartheta_0)[(m+n)^2-\vartheta_0]}\right.\right.$$
$$\left.\left.+\frac{(m^2-mn+n^2-3\vartheta_0)\mathrm{Re}[\vartheta_m\vartheta_n^* e^{i2(m-n)\pi s/L}]}{(m^2-\vartheta_0)(n^2-\vartheta_0)[(m-n)^2-\vartheta_0]}\right]\right\}. \tag{30}$$

Note that β_2 contributes a constant term

$$\sum_{n=1}^{\infty}|\vartheta_n|^2\frac{\sqrt{\vartheta_0}\pi(n^2-\vartheta_0)\cot(\sqrt{\vartheta_0}\pi)-(n^2-3\vartheta_0)}{4\,\vartheta_0(n^2-\vartheta_0)^2}, \tag{31}$$

in the square bracket of Eq. (29). The constant term in $\beta(s)$ could be important for optimization since it determines the average β value.

III. APPLICATION EXAMPLES

The possibility of muon colliders or neutrino factories has received much attention recently [5,12,13]. The biggest challenge is to reduce the beam emittance (cooling) to a useful level before a significant amount of muons decay. Ionization energy loss of muons in materials is the only known process that is sufficiently fast for this purpose. However, multiple scattering tends to enlarge beam emittance (heating) and its effect is proportional to the beam size. Thus, a muon beam has to be tightly focused during cooling. Channels with continuous solenoidal focusing are the major candidates. Since the canonical angular momentum will build up in a unipolar solenoid cooling channel, the on-axis field polarity must be changed [14,12]. One type of design is to have the field periodically switched. A natural thought along this line is the ''FOFO'' channel, whose on-axis field varies sinusoidally [15]. In order to reach smaller beta function at absorbers and

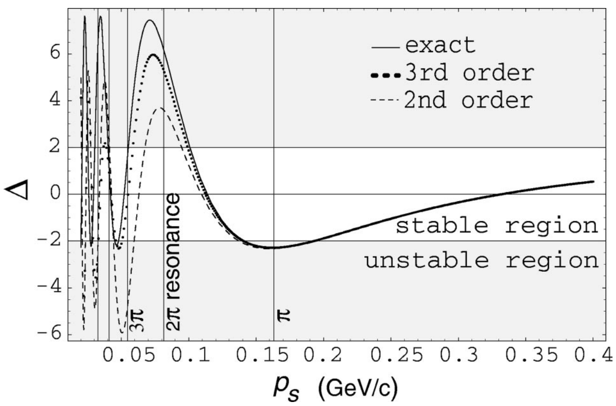

FIG. 1. Stability boundaries of a ''SuperFOFO'' channel.

larger momentum acceptance, variations are derived from the simple FOFO channel. Here, two examples are discussed briefly to show the accuracy of our approximate formulas. Understanding of channel behavior obtained from analytical analysis is mentioned; however, it is not our focus here, and thus, not meant to be comprehensive.

For periodic solenoid channels

$$\vartheta_0 = \left(\frac{qL}{2\pi p_s}\right)^2 \langle B_s^2\rangle_{\text{one-period}}. \tag{32}$$

Here, the angular brackets denote taking the average over one period. From Eqs. (26), (32), the width of the unstable region, *stopband*, in relative momentum is about $|\vartheta_n/\vartheta_0|$. For a FOFO channel, the focusing function $K(s) = (qB_{\max}/2p_s)^2\sin^2(\pi s/L)\sim 1-\cos(2s)$, thus the only harmonic content is $|\vartheta_{\pm 1}/\vartheta_0| = 1/2$. This is large and results in small momentum acceptance. To improve performance, more harmonics are added.

In a ''superFOFO'' channel considered at Lawrence Berkeley National Laboratory [16], $B_s(s) = B_{\max}/1.7[\sin(\pi s/L) + 1.1\sin(5\pi s/L) + 0.2\sin(5\pi s/L)]$ and its Fourier coefficients ϑ_n at $p_s = 100$ MeV/c are $\{\vartheta_0, \vartheta_{\pm 1}, \ldots, \vartheta_{\pm 5}\} = \{2.681, 0.977, -1.072, -0.959, -0.262, -0.024\}$. Scaling the coefficients by $1/p_s^2$, in Fig. 1 we plot Δ as a function of p_s in various approximations. The figure exhibits the stability

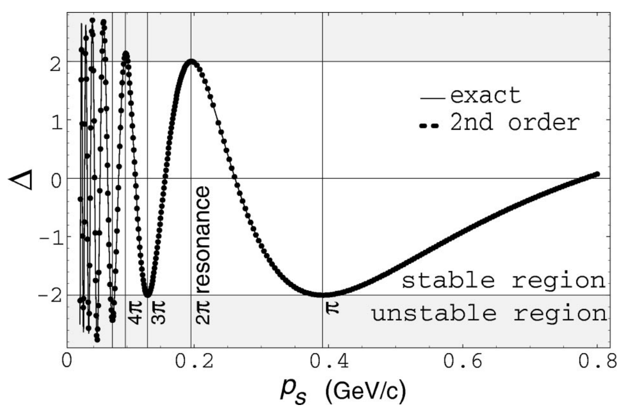

FIG. 2. Stability boundaries of a fast-field-flip solenoidal channel.

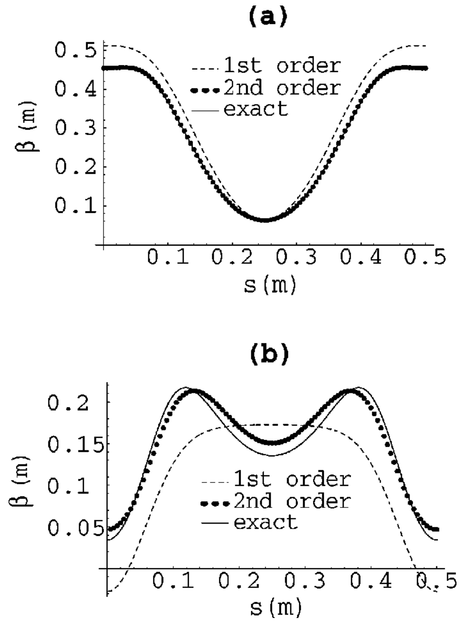

FIG. 3. β functions of a SuperFOFO channel in (a) the first and (b) the second passbands.

boundaries and reveals the accuracy of the approximate formulas. We also indicated the unperturbed location ($\sqrt{\vartheta_0} = n$) of the $n\pi$ resonance with vertical lines. Note that the third-order approximation of Δ reproduces reasonably well the particularly interesting first two passbands at $p_s \geqslant 0.2$ and 0.1 GeV/c$\leqslant p_s \leqslant 0.14$ GeV/c. In the ''fast-field-flip'' channel proposed by Balbekov [17], the coefficients (for a 200 MeV/c muon and magnetic period of 5.19 m) are $\{\vartheta_0, \vartheta_{\pm 1}, \ldots, \vartheta_{\pm 14}\} = \{3.8186, -0.0039, -0.0047, -0.0060, -0.2239, -0.3203, -0.3345, -0.2976, -0.2383, -0.1759, -0.1212, -0.0783, -0.0477, -0.0272, -0.0146\}$. Note that the higher-order coefficients are strongly suppressed and thus yield better momentum acceptance. The corresponding stability calculation is shown in Fig. 2. Notice the second-order formula Eq. (22) works extremely well due to the weakness of the harmonics. The figure demonstrates clearly that the nth stopband width is proportional to $|\vartheta_n|$. Due to the smallness of the first three harmonics, the π resonance barely exists, and the 2π and 3π resonances are completely suppressed.

Optimizing the envelope function $\beta(s)$ by a suitable arrangement of focusing elements is one of the most important

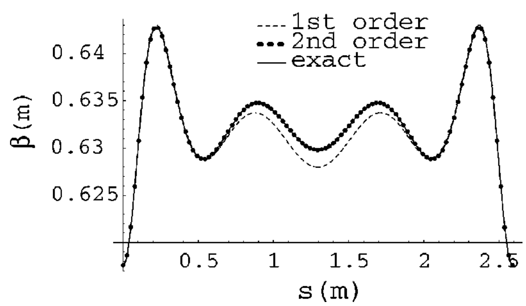

FIG. 4. β function of a fast-field-flip solenoidal channel.

tasks in accelerator design. The simplicity of Eq. (29) should make it a useful tool for this task. In order to reduce the beam size, the $\beta(s)$ should in general be small. This requirement is especially critical for the ionization cooling of muon beams: the overall β value should be small in order to be able to transport the large emittance beam; the minimum β should be at the absorber locations and be as small as possible in order to minimize the heating due to multiple scattering [18,12]. The form of Eq. (29) already suggests a few general strategies for obtaining a small $\beta(s)$: (1) shorter period; (2) stronger field (resulting in larger ϑ_0 and higher-order passband); (3) larger phase-advance term $\sin\mu$; and (4) larger harmonics to cancel the unity term within the brackets of Eq. (29). Furthermore, it is helpful to make Eq. (31) negative and as large as possible. Note that these strategies are often in conflict with the constraint of orbit stability and available magnetic field strength.

Figures 3(a) and 3(b) are plots of the β functions for the superFOFO channel in the first and second passbands ($p_s = 200$ and 120 MeV), respectively. We see that the second-order formula gives a reasonable approximation in this case. Figure 4 shows a similar plot for the case of the fast-flip channel in the second momentum passband ($p_s = 300$ MeV). Due to the suppression of harmonics, the first-order formula is already quite accurate.

We have demonstrated the applicability of the approximate formulas to solenoid channels for muon ionization cooling, obtained insight into a channel's performance, and provided useful design guidelines. However, reaching a good design is much more involved and is a subject beyond this paper.

IV. CONCLUSION

To conclude, we developed a systematic treatment of beam dynamics in periodic focusing channels. A general formula is given for the β function. Also we have rederived approximate formulas for the trace of the one-period transfer matrix, which exist in the mathematical literature but have not been appreciated in the beam physics community. Formulas for other quantities can be derived similarly. These results, especially Eqs. (22) and (29), should be important for analyzing and optimizing periodic focusing channels that are the basic building blocks in accelerators, particularly for cases where thin lens approximation fails.

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APPENDIX

If we let $\omega=\sqrt{\vartheta_0}$, Eq. (21) gives

$$\Delta_2=\frac{1}{2\omega^2}\sum_{m,n=-\infty}^{\infty}\tilde{\vartheta}_m\tilde{\vartheta}_n\int_0^{\pi}d\varsigma\int_0^{\varsigma}d\bar{\varsigma}\,e^{i2n\varsigma}e^{i2m\bar{\varsigma}}\times[\cos\omega\pi-\cos\omega(\pi-2\varsigma+2\bar{\varsigma})].$$

The first term, containing $\cos\omega\pi$, becomes

$$\frac{\cos\omega\pi}{2\omega^2}\int_0^{\pi}d\varsigma\int_0^{\varsigma}d\bar{\varsigma}\sum_{m,n=-\infty}^{\infty}\tilde{\vartheta}_m\tilde{\vartheta}_n e^{i2(n\varsigma+m\bar{\varsigma})}$$
$$=\frac{\cos\omega\pi}{4\omega^2}\int_0^{\pi}d\varsigma\int_0^{\pi}d\bar{\varsigma}\sum_{m,n=-\infty}^{\infty}\tilde{\vartheta}_m\tilde{\vartheta}_n e^{i2(n\varsigma+m\bar{\varsigma})}\sim|\tilde{\vartheta}_0|^2$$
$$=0.$$

Here, the change of the integration limit is valid since the integrand is symmetric about ς and $\bar{\varsigma}$.

The second term yields

$$\Delta_2=I(\omega)+I(-\omega),$$

where

$$I(\omega)=-\frac{e^{i\omega\pi}}{4\omega^2}\sum_{m,n=-\infty}^{\infty}\tilde{\vartheta}_m\tilde{\vartheta}_n\int_0^{\pi}d\varsigma\int_0^{\varsigma}d\bar{\varsigma}\,e^{i2(n-\omega)\varsigma}e^{i2(m+\omega)\bar{\varsigma}}$$
$$=-\frac{e^{i\omega\pi}}{4\omega^2}\sum_{m,n=-\infty}^{\infty}\frac{\tilde{\vartheta}_m\tilde{\vartheta}_n}{2i(m+\omega)}\int_0^{\pi}d\varsigma[e^{i2(m+n)\varsigma}-e^{i2(n-\omega)\varsigma}].$$

The first integrand gives zero unless $m+n=0$, while the second integrand leads to an asymmetric form in ω and thus does not contribute to Δ_2. Therefore, we have

$$\Delta_2=\frac{\pi}{4\omega^2}\sum_{n=-\infty}^{\infty}|\tilde{\vartheta}_n|^2\frac{1}{2i}\left[\frac{e^{i\omega\pi}}{n-\omega}+\frac{e^{-i\omega\pi}}{n+\omega}\right]$$
$$=\frac{\pi}{4\omega^2}\sum_{n=-\infty}^{\infty}\frac{|\tilde{\vartheta}_n|^2}{n^2-\omega^2}[\omega\sin\omega\pi-i\,n\cos\omega\pi].$$

The last term sums to zero since it is odd in n and $\tilde{\vartheta}_0=0$. Thus, we obtain

$$\Delta_2=\frac{\pi\sin\omega\pi}{2\omega}\sum_{n=1}^{\infty}\frac{|\tilde{\vartheta}_n|^2}{n^2-\omega^2},$$

which is the second term in Eq. (22).

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[10] At the stability boundaries given by the equal sign of Eq. (6), the orbits may or may not be stable. For accelerator applications, the stability on the boundaries is not so important since particles need to be away from unstable regions.
[11] Please note that Eqs. (23) and (24) provide an efficient numeric method to compute Δ as accurately as necessary by using a sufficiently large matrix. In the case of $\vartheta_0 \sim (2n)^2$, one may use the equivalent but different expression (Ref. [9])

$$\Delta = -2 + 4\cos^2\left(\frac{\pi}{2}\sqrt{\vartheta_0}\right)\left\| \frac{\tilde{\vartheta}_{n-m}}{\vartheta_0 - (2n+1)^2} + \delta_{nm} \right\|_{-\infty}^{\infty}.$$

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